

CHARACTERISTIC FACTORS FOR COMMUTING ACTIONS OF AMENABLE GROUPS

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ABSTRACT. We describe characteristic factors for certain averages arising from commuting actions of locally compact, second-countable, amenable groups. Under some ergodicity assumptions we use these factors to prove a form of multiple recurrence for three such actions.

1. INTRODUCTION

Furstenberg and Katznelson's multiple recurrence theorem [FK78] states that if T_1, \dots, T_k are commuting, measure-preserving transformations of a probability space (X, \mathcal{B}, μ) then

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(B \cap T_1^{-n} B \cap \dots \cap T_k^{-n} B) > 0$$

for any B in \mathcal{B} with $\mu(B) > 0$. It is natural to ask whether such a result holds for commuting actions of groups, by which we mean actions T_1, \dots, T_k of a group G on a probability space (X, \mathcal{B}, μ) by measure-preserving transformations that satisfy $T_i^g T_j^h = T_j^h T_i^g$ for all $g, h \in G$ and all $1 \leq i < j \leq k$. Unfortunately the results in [BH92] suggest that in certain cases

$$\mu(B \cap (T_1^g)^{-1} B \cap \dots \cap (T_k^g)^{-1} B) = 0$$

for all $g \neq 1$ in G . However, if one instead considers multiple recurrence of the form

$$\mu(B \cap (T_k^g \dots T_1^g)^{-1} B \cap (T_k^g \dots T_2^g)^{-1} B \cap \dots \cap (T_k^g)^{-1} B) > 0$$

then the situation is more promising. Bergelson, McCutcheon and Zhang [BMZ97] proved that when G is countable and amenable and $\mu(B) > 0$ the set

$$\{g \in G : \mu(B \cap (T_2^g T_1^g)^{-1} B \cap (T_2^g)^{-1} B) > 0\} \quad (1.1)$$

is syndetic, meaning that finitely many of its left-shifts cover G . In fact, Bergelson and McCutcheon [BM07] have shown for any countable group G that (1.1) belongs to any minimal idempotent ultrafilter in βG . Also, it follows from the work of Bergelson and Rosenblatt (Theorem 2.4 in [BR88]) that if G is amenable and if $T_j \dots T_i$ is weakly-mixing for all $1 \leq i \leq j \leq k$ then

$$\{g \in G : \mu(B \cap (T_k^g \dots T_1^g)^{-1} B \cap \dots \cap (T_k^g)^{-1} B) \geq \mu(B)^{k+1}\}$$

has full density with respect to any Følner sequence in G . More generally, Bergelson has made the following conjecture.

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Conjecture 1.1 (Section 5 of [Ber96]). *Let G be a countable amenable group with a left Følner sequence Φ . Let T_1, \dots, T_k be commuting, measure-preserving actions of G on a probability space (X, \mathcal{B}, μ) . Then*

$$\liminf_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \mu(B \cap (T_k^g \cdots T_1^g)^{-1} B \cap \cdots \cap (T_k^g)^{-1} B) > 0 \quad (1.2)$$

for any $B \in \mathcal{B}$ with $\mu(B) > 0$.

In this paper we describe characteristic factors for the average

$$\frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \prod_{i=1}^k T_k^g \cdots T_i^g f_i \quad (1.3)$$

that will allow us to verify a version of Bergelson's conjecture when $k = 3$ under the assumption that the actions T_1, T_2 and $T_2 T_1$ are ergodic. (In fact we will do so for locally-compact, second-countable, amenable groups, but only discuss the discrete case in the introduction.) By *characteristic factors* we mean $T_k \cdots T_i$ invariant sub- σ -algebras $\mathcal{C}_{k,i}$ of \mathcal{B} such that

$$\lim_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \left(\prod_{i=1}^k T_k^g \cdots T_i^g f_i - \prod_{i=1}^k T_k^g \cdots T_i^g \mathbb{E}(f_i | \mathcal{C}_{k,i}) \right) = 0$$

in $L^2(X, \mathcal{B}, \mu)$ for any f_i in $L^\infty(X, \mathcal{B}, \mu)$. By identifying characteristic factors we reduce the study of the limiting behaviour of (1.3) to the situation where f_i is $\mathcal{C}_{k,i}$ measurable.

Although the terminology is more recent, this technique was first used by Furstenberg in his ergodic proof [Fur77] of Szemerédi's theorem. Therein he exhibited, for any ergodic, measure-preserving transformation T of a probability space (X, \mathcal{B}, μ) , an increasing sequence \mathcal{Z}_k of T invariant sub- σ -algebras, with \mathcal{Z}_k an isometric extension of \mathcal{Z}_{k-1} , such that

$$\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int \prod_{i=1}^k T^{in} f_i \cdot f_{k+1} - \prod_{i=1}^k T^{in} \mathbb{E}(f_i | \mathcal{Z}_k) \cdot f_{k+1} d\mu = 0$$

in $L^2(X, \mathcal{B}, \mu)$ for any f_i in $L^\infty(X, \mathcal{B}, \mu)$. Furstenberg then used the properties of isometric extensions to show by induction on i that

$$\liminf_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(B \cap T^{-n} B \cap \cdots \cap T^{-kn} B) > 0 \quad (1.4)$$

for any B in \mathcal{Z}_i having positive measure.

More recently, Host and Kra [HK05] and Ziegler [Zie07] have shown that one can replace \mathcal{Z}_k with a smaller sub- σ -algebra that corresponds to an inverse limit of k -step nilrotations. This has led to sharper (e.g. [BHK05], [BLL08]) combinatorial results. Also, Frantzikinakis and Kra [FK05] have shown, under natural ergodicity assumptions, that inverse limits of commuting rotations on a nilmanifold are characteristic for commuting \mathbb{Z} actions.

Our techniques are similar to those used in [Fur77]. However, since we deal with commuting actions, our characteristic factors are more complicated: we will show inductively that $\mathcal{C}_{k,i}$ is a $T_k \cdots T_i$ compact extension of $\mathcal{C}_{k-1,i}$ for each $1 \leq i \leq k-1$, and that $\mathcal{C}_{k,k}$ is a T_k almost-periodic extension of $\mathcal{C}_{k-1} = \mathcal{C}_{k-1,1} \vee \cdots \vee \mathcal{C}_{k-1,k-1}$.

(See Figure 1 on Page 14 for a schematic.) It is not clear whether these characteristic factors can be used to prove Bergelson's conjecture. The difficulty lies partly in their dependence on i which, as exemplified in [Zha95], cannot be removed in general. Under the above-mentioned ergodicity assumptions we can handle this dependence when $k = 3$ and obtain multiple recurrence.

The rest of the paper runs as follows. In the next two sections we recall definitions and results used throughout the remainder of the paper. In Section 4 we prove some facts about almost-periodic functions and eigenfunctions over a factor that we will need to prove our factors are characteristic. Section 5 contains a definition of the factors $\mathcal{C}_{k,i}$ and a proof that they are characteristic. The following section contains a result that allows us to lift multiple recurrence from a single σ -algebra to a family of σ -algebras. It is used in Section 7 to prove our multiple recurrence result. Finally, we present some further consequences of our description of characteristic factors, including some combinatorial results, in Section 8.

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2. PRELIMINARIES

In this section we recall the facts we will need about measurable group actions, factors, disintegration of measures, joinings and IP^* sets. We also give suitable versions of the van der Corput trick and the mean ergodic theorem. For more details, see [Fur77], [Fur81] and [Gla03].

Fix throughout this paper a locally-compact, second-countable, amenable group G with a left Haar measure m and a countable, dense subgroup Γ . Amenability implies (4.16 in [Pat88]) the existence of a sequence Φ of compact, positive-measure subsets of G such that

$$\frac{m(\Phi_N \triangle g\Phi_N)}{m(\Phi_N)} \rightarrow 0$$

as $N \rightarrow \infty$ for each $g \in G$. The convergence is uniform on compact subsets of G . Any such sequence is called a *(left) Følner sequence*. Fix a left Følner sequence Φ in G .

Let (X, \mathcal{B}, μ) be a separated, countably generated probability space. By a *measurable action* of G on such a space we mean a family $\{T^g : g \in G\}$ of measurable, measure-preserving transformations of (X, \mathcal{B}, μ) such that the induced map $G \times X \rightarrow X$ given by $(g, x) \mapsto T^g x$ is measurable and $T^g T^h = T^{gh}$ for all g, h in G . Two such actions T_1 and T_2 are said to *commute* if $T_1^g T_2^h = T_2^h T_1^g$ for all g, h in G , and if they do $T_1^g T_2^g$ is also a measurable action of G on (X, \mathcal{B}, μ) .

By a *system* we mean a tuple (X, \mathcal{B}, μ, T) consisting of a measurable action T of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . We often write \mathbf{X} for (X, \mathcal{B}, μ, T) and $L^p(\mathbf{X})$ for the corresponding real space $L^p(X, \mathcal{B}, \mu)$. Given a system \mathbf{X} , each T^g induces a unitary operator on $L^2(\mathbf{X})$ given by $(T^g f)(x) = f(T^g x)$. It is immediate that $T^g(T^h f) = T^{hg} f$ for all g, h in G . Since \mathcal{B} is countably generated the Hilbert space $L^2(\mathbf{X})$ is separable. By 22.20(b) in [HR79]

and the fact that $G \times X \rightarrow X$ is measurable, the map $g \mapsto T^g$ is continuous in the strong operator topology.

Given a sub- σ -algebra \mathcal{C} of \mathcal{B} and f in $L^2(X, \mathcal{B}, \mu)$ the *conditional expectation* of f on \mathcal{C} , denoted $\mathbb{E}(f|\mathcal{C})$, is the orthogonal projection of f onto the closed subspace $L^2(X, \mathcal{C}, \mu)$. We say that a sub- σ -algebra \mathcal{C} of \mathcal{B} is *T invariant* if $(T^g)^{-1}C \in \mathcal{C}$ for all $C \in \mathcal{C}$ and all $g \in G$. When this is the case each T^g commutes with the conditional expectation $\mathbb{E}(\cdot|\mathcal{C})$.

We say that a system $\mathbf{Y} = (Y, \mathcal{D}, \lambda, S)$ is a *factor* of $\mathbf{X} = (X, \mathcal{B}, \mu, T)$, or that \mathbf{X} is an *extension* of \mathbf{Y} , if there is a measurable, measure-preserving map $\pi : X \rightarrow Y$, called the *factor map*, that intertwines the actions T and S , meaning that $\pi(T^g x) = S^g(\pi x)$ for all x in X and all g in G . We will usually abuse notation by writing μ for λ and T for S . To any factor \mathbf{Y} of \mathbf{X} we can associate the T -invariant sub- σ -algebra $\pi^{-1}\mathcal{D}$ of \mathcal{B} . We can use π to identify $L^2(\mathbf{Y})$ with $L^2(X, \pi^{-1}\mathcal{D}, \mu, T)$ isometrically. This lets us think of $\mathbb{E}(f|\pi^{-1}\mathcal{D})$ as an element of $L^2(\mathbf{Y})$, which we will denote $\mathbb{E}(f|\mathbf{Y})$.

By Lemma 3.1 in [FK91] any closed subspace of $L^2(\mathbf{X})$ that is a lattice and contains the constants is of the form $L^2(X, \mathcal{C}, \mu)$ for some sub- σ -algebra \mathcal{C} of \mathcal{B} . If the subspace is T -invariant then so is \mathcal{C} . Proposition 2.1 in [Zim76] lets us associate with any T -invariant sub- σ -algebra \mathcal{C} of \mathcal{B} a system \mathbf{Y} and a T -invariant, full-measure set X' in \mathcal{B} such that $(X', \mathcal{B}, \mu, T)$ is an extension of \mathbf{Y} via a factor map $\pi : X' \rightarrow Y$. Since the probability space defined by X' is also separated and countably generated, we will not distinguish between X' and X hereafter.

A factor map $\pi : \mathbf{X} \rightarrow \mathbf{Y}$ gives rise to a *disintegration* of μ over \mathbf{Y} , which is a λ almost-surely defined family $\{\mu_y : y \in Y\}$ of probability measures on (X, \mathcal{B}) with the following properties.

- (1) For any \mathcal{B} -measurable function f that is square-integrable the map

$$y \mapsto \int f d\mu_y$$

is defined λ almost-surely and \mathcal{D} -measurable.

- (2) For any \mathcal{B} -measurable function f that is square-integrable

$$\mathbb{E}(f|\mathbf{Y})(y) = \int f d\mu_y$$

λ almost-surely.

- (3) The group G permutes the family μ_y in the sense that, for any $g \in G$ and any \mathcal{B} -measurable function f that is square-integrable one has

$$\int T^g f d\mu_y = \int f d\mu_{S^g y}$$

λ almost-surely.

Care is taken to speak of a function $f : X \rightarrow \mathbb{R}$ rather than an equivalence class of functions in $L^2(X, \mathcal{B}, \mu)$ because the measures μ_y may well be singular with respect to μ . Although each integrable function $f : X \rightarrow \mathbb{R}$ defines an equivalence class in the space $L^2(X, \mathcal{B}, \mu_y)$ for almost every y , changing f on a set of μ measure zero may not preserve all of these classes. Write $\langle \cdot, \cdot \rangle_y$ for the inner product on $L^2(X, \mathcal{B}, \mu_y)$ and $\| \cdot \|_y$ for the corresponding norm. Given an invariant sub- σ -algebra \mathcal{D} , by the disintegration of μ over \mathcal{D} we mean the family of measure $\nu_x = \mu_{\pi x}$ where μ_y is an

almost-surely defined disintegration of μ over a factor corresponding to \mathcal{D} . By an abuse of notation we will write μ_x for ν_x .

We now recall some basic facts about joinings. Let $(X_i, \mathcal{B}_i, \mu_i), 1 \leq i \leq k$ be probability spaces and let π_i be the projection from $X_1 \times \cdots \times X_k$ to X_i . We say that a probability measure ν on $(X_1 \times \cdots \times X_k, \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_k)$ is a *standard measure* if $\nu(\pi_i^{-1}B) = \mu_i(B)$ for all $B \in \mathcal{B}_i$ and all $1 \leq i \leq k$. A sequence ν_n of standard measures is said to converge to a standard measure ν if

$$\nu_n(B_1 \times \cdots \times B_k) \rightarrow \nu(B_1 \times \cdots \times B_k)$$

for all B_i in \mathcal{B}_i . A *joining* of systems $\mathbf{X}_1, \dots, \mathbf{X}_k$ is any system $\mathbf{X} = (X, \mathcal{B}, \nu, T)$ where $X = X_1 \times \cdots \times X_k$, $\mathcal{B} = \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_k$, $T^g = T_1^g \times \cdots \times T_k^g$ and ν is a standard measure that is T -invariant. Given a factor \mathbf{Y}_i of \mathbf{X}_i for each i , we can consider the system $\mathbf{Y} = (Y, \mathcal{D}, \eta, T)$ made from \mathbf{X} by projecting ν onto the product (Y, \mathcal{D}) of the underlying measurable spaces (Y_i, \mathcal{D}_i) . Call a joining \mathbf{X} of the \mathbf{X}_i a *conditional product joining* relative to the factors \mathbf{Y}_i if

$$\int f_1 \otimes \cdots \otimes f_k d\nu = \int \mathbb{E}(f_1 | \mathbf{Y}_1) \otimes \cdots \otimes \mathbb{E}(f_k | \mathbf{Y}_k) d\eta \quad (2.1)$$

for all f_i in $L^\infty(\mathbf{X}_i)$. Here $f_1 \otimes \cdots \otimes f_k$ denotes the function mapping (x_1, \dots, x_k) to $f_1(x_1) \cdots f_k(x_k)$. We can re-write (2.1) as

$$\nu = \int \mu_{1, y_1} \otimes \cdots \otimes \mu_{k, y_k} d\eta(y_1, \dots, y_k) \quad (2.2)$$

if μ_{i, y_i} is the almost-surely defined disintegration of μ_i over \mathbf{Y}_i .

Let T_1, \dots, T_k be commuting, measurable actions of G on a probability space (X, \mathcal{B}, μ) . Define a measure ν_k on $(X^{k+1}, \mathcal{B}^{k+1})$ by

$$\int f_1 \otimes \cdots \otimes f_{k+1} d\nu_k = \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int f_{k+1} \cdot \prod_{i=1}^k T_k^g \cdots T_i^g f_i d\mu dm(g) \quad (2.3)$$

for any f_1, \dots, f_{k+1} in $L^\infty(X, \mathcal{B}, \mu)$. The existence of the limit is justified by Theorem 1.3 in [ZK11]. Using the fact that Φ is a Følner sequence, one can show that ν_k is

$$T_k T_{k-1} \cdots T_1 \times \cdots \times T_k T_{k-1} \times T_k \times I$$

invariant. Thus the measure ν_k yields a joining of the systems

$$(X, \mathcal{B}, \mu, T_k \cdots T_1), \dots, (X, \mathcal{B}, \mu, T_k), (X, \mathcal{B}, \mu, I)$$

called the *Furstenberg joining* of the actions T_1, \dots, T_k .

Given two systems $\mathbf{X}_1 = (X_1, \mathcal{B}_1, \mu_1, T_1)$ and $\mathbf{X}_2 = (X_2, \mathcal{B}_2, \mu_2, T_2)$ having a common factor $\mathbf{Y} = (Y, \mathcal{D}, \mu, T)$ via factor maps π_1 and π_2 respectively, we can form their *relatively independent joining*

$$\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2 = (X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \nu, T_1 \times T_2)$$

where ν is the measure defined by

$$\int f_1 \otimes f_2 d\nu = \int \mathbb{E}(f_1 | \mathbf{Y}) \cdot \mathbb{E}(f_2 | \mathbf{Y}) d\mu$$

for all f_1 in $L^\infty(\mathbf{X}_1)$ and all f_2 in $L^\infty(\mathbf{X}_2)$. The measure is supported on the set

$$\{(x_1, x_2) : \pi_1 x_1 = \pi_2 x_2\}$$

so \mathbf{Y} is a factor of $\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2$ in an unambiguous way. If $\mu_{1,y}$ and $\mu_{2,y}$ are the almost-surely defined disintegrations of μ_1 and μ_2 over \mathbf{Y} then

$$\nu = \int \mu_{1,y} \otimes \mu_{2,y} d\mu(y) \quad (2.4)$$

is the disintegration of ν over \mathbf{Y} . We also recall that

$$\nu = \int \mu_{1,\pi_2 x_2} \otimes \delta_{x_2} d\mu_2(x_2) \quad (2.5)$$

is the disintegration of ν over \mathbf{X}_2 .

We will need some basic facts about IP sets. Given a sequence ϕ in G define

$$\text{FP}(\phi) = \{\phi(i_1) \cdots \phi(i_k) : k \in \mathbb{N}, i_1 < \cdots < i_k \in \mathbb{N}\}$$

and call a subset of G an *IP set* if it contains $\text{FP}(\phi)$ for some sequence ϕ in G . By Hindman's theorem (see Lemma 2.1 in [BH93]) the property of being an IP set is partition regular. A subset of G is said to be IP^* if its intersection with every IP set is non-empty. It follows from partition regularity (see Lemma 9.5 in [Fur81]) that the intersection of two IP^* sets is also IP^* . Finally, note that every IP^* subset of G has the property that finitely many of its left-shifts cover G . This is because the complement of a set failing to have this property contains a right-shift of any finite set and therefore contains an IP set. Thus every measurable IP^* set has positive lower density with respect to Φ , where

$$\underline{d}_{\Phi}(E) = \liminf_{N \rightarrow \infty} \frac{m(E \cap \Phi_N)}{m(\Phi_N)}$$

is the *lower density* of a measurable subset E of G with respect to Φ . Replacing \liminf with \limsup gives the *upper density* of E , denoted $\overline{d}(E)$, and when $\overline{d}(E) = \underline{d}(E)$ their common value, the *density* of E , is denoted $d(E)$.

We conclude this section with versions of the van der Corput trick and the mean ergodic theorem suitable for our needs. The Hilbert space valued integrals below are always taken in the sense of Bochner.

Proposition 2.1 (van der Corput trick). *Let \mathcal{H} be a separable Hilbert space and let $u : G \rightarrow \mathcal{H}$ be weakly measurable and uniformly bounded in norm. If*

$$\limsup_{H \rightarrow \infty} \frac{1}{m(\Phi_H)^2} \int_{\Phi_H} \int_{\Phi_H} \limsup_{N \rightarrow \infty} \left| \frac{1}{m(\Phi_N)} \int_{\Phi_N} \langle u(hg), u(lg) \rangle dm(g) \right| dm(h) dm(l) = 0$$

then

$$\left\| \frac{1}{m(\Phi_N)} \int_{\Phi_N} u(g) dm(g) \right\| \quad (2.6)$$

converges to 0 as $N \rightarrow \infty$.

Proof. Fix $\varepsilon > 0$. First note that given any H in \mathbb{N} one has

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{m(\Phi_N)} \int_{\Phi_N} u(g) dm(g) - \frac{1}{m(\Phi_N)} \int_{\Phi_N} \frac{1}{m(\Phi_H)} \int_{\Phi_H} u(hg) dm(h) dm(g) \right\| = 0$$

by the dominated convergence theorem and the fact that Φ is a left Følner sequence. By the Cauchy-Schwarz inequality

$$\begin{aligned} & \left\| \frac{1}{m(\Phi_N)} \int_{\Phi_N} \frac{1}{m(\Phi_H)} \int_{\Phi_H} u(hg) \, dm(h) \, dm(g) \right\|^2 \\ & \leq \frac{1}{m(\Phi_N)} \int_{\Phi_N} \left\| \frac{1}{m(\Phi_H)} \int_{\Phi_H} u(hg) \, dm(h) \right\|^2 dm(g) \\ & = \frac{1}{m(\Phi_H)^2} \int_{\Phi_H} \int_{\Phi_H} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \langle u(hg), u(kg) \rangle dm(g) dm(h) dm(k) \end{aligned}$$

which allows us to relate (2.6) to the hypothesis and obtain the desired result. \square

Proposition 2.2 (Mean ergodic theorem). *Let T be a measurable action of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Let \mathcal{I} be the sub- σ -algebra of T -invariant sets. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} T^g f \, dm(g) = \mathbb{E}(f | \mathcal{I})$$

in norm for all f in $L^2(X, \mathcal{B}, \mu)$.

This is Theorem 5.7 in [Pat88]. In particular we have

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \int T^g f_1 \cdot f_2 \, d\mu \, dm(g) = \int \mathbb{E}(f_1 | \mathcal{I}) \cdot \mathbb{E}(f_2 | \mathcal{I}) \, d\mu$$

for all f_1, f_2 in $L^2(X, \mathcal{B}, \mu)$.

3. BOREL HILBERT BUNDLES

In this section we recall how to associate Borel Hilbert bundles with extensions and relatively independent joinings. For details, see [Dix81], [Gla03] and [Wil07].

Let (Y, \mathcal{D}, μ) be a separable, countably generated probability space and let $\mathfrak{H} = \{\mathfrak{H}_y : y \in Y\}$ be a collection of separable, real Hilbert spaces. Write $\langle \cdot, \cdot \rangle_y$ for the inner product on \mathfrak{H}_y . From Y and \mathfrak{H} we can form the total space $Y * \mathfrak{H} = \{(y, h) : y \in Y, h \in \mathfrak{H}_y\}$ which comes with a projection $\pi : Y * \mathfrak{H} \rightarrow Y$. The spaces \mathfrak{H}_y are called the *fibers* of the total space. A *section* of $Y * \mathfrak{H}$ is any map $f : Y \rightarrow Y * \mathfrak{H}$ such that $\pi \circ f$ is the identity. The image of a point y under a section f is a point in $Y * \mathfrak{H}$ which we will write as (y, f_y) . Thus f_y belongs to \mathfrak{H}_y . To any section f we can associate the map $\tilde{f} : Y * \mathfrak{H} \rightarrow \mathbb{R}$ defined by $\tilde{f}(y, h) = \langle f_y, h \rangle_y$. A *Borel Hilbert bundle* is a Hilbert bundle $Y * \mathfrak{H}$ equipped with a σ -algebra of subsets of $Y * \mathfrak{H}$ for which:

- (i) the projection $Y * \mathfrak{H} \rightarrow Y$ is measurable;
- (ii) there is a sequence \tilde{f}_n of sections such that:
 - (a) the maps \tilde{f}_n are measurable;
 - (b) for each n, m the map $Y \rightarrow \mathbb{R}$ given by $y \mapsto \langle f_{n,y}, f_{m,y} \rangle_y$ is measurable;
 - (c) the functions \tilde{f}_n and π separate points on $Y * \mathfrak{H}$.

To associate a Borel Hilbert bundle $Y * \mathfrak{H}$ with a given extension $\mathbf{X} \rightarrow \mathbf{Y}$, fix an almost-surely defined disintegration μ_y of μ over \mathbf{Y} and let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a countable, Γ -invariant sub-algebra of \mathcal{B} that generates \mathcal{B} . For each n, m the function $y \mapsto \langle 1_{A_n}, 1_{A_m} \rangle_y$ is defined on a full-measure subset of Y and is measurable there. Let Y_0 be a Γ -invariant, full-measure subset of Y on which μ_y and all of the functions $y \mapsto \langle 1_{A_n}, 1_{A_m} \rangle_y$ are defined and on which $T^\gamma \mu_y = \mu_{T^\gamma y}$ for all γ in Γ . Put $\mathfrak{H}_y = L^2(X, \mathcal{B}, \mu_y)$ when $y \in Y_0$ and put $\mathfrak{H}_y = \{0\}$ otherwise. Each \mathfrak{H}_y is separable because \mathcal{B} is countably generated. Let \mathfrak{H} be the collection $\{\mathfrak{H}_y : y \in Y\}$. Define a sequence f_n of sections by taking $f_{n,y} = 1_{A_n}$ when $y \in Y_0$ and $f_{n,y} = 0$ otherwise. Equip $Y * \mathfrak{H}$ with the smallest σ -algebra of subsets for which π and the maps \tilde{f}_n are measurable. It is immediate from the construction that this σ -algebra makes $Y * \mathfrak{H}$ into a Borel Hilbert bundle. Moreover, a section $f : Y \rightarrow Y * \mathfrak{H}$ is measurable with respect to this σ -algebra if and only if $y \mapsto \langle f_y, f_{n,y} \rangle_y$ is measurable for each n . We call $Y * \mathfrak{H}$ the Borel Hilbert bundle corresponding to the extension $\mathbf{X} \rightarrow \mathbf{Y}$. The Hilbert space $L^2(Y * \mathfrak{H}, \mu)$ formed from the set

$$\mathcal{L}^2(Y * \mathfrak{H}, \mu) = \{f \in B(Y * \mathfrak{H}) : y \mapsto \|f_y\|_y^2 \text{ is } \mu \text{ integrable}\}$$

of square-integrable sections by identifying sections that agree almost surely is isomorphic to $L^2(X, \mathcal{B}, \mu)$. Thus to any ϕ in $L^2(\mathbf{X})$ we can associate an almost-surely defined, square-integrable section $y \mapsto \phi_y$ and vice versa.

We now recall how Γ acts on sections of $Y * \mathfrak{H}$. Fix $\gamma \in \Gamma$. Since $T^\gamma \mu_y = \mu_{T^\gamma y}$ whenever $y \in Y_0$ the map $T_y^\gamma : \mathfrak{H}_{T^\gamma y} \rightarrow \mathfrak{H}_y$ given by $(T_y^\gamma f)(x) = f(T^\gamma x)$ is well-defined and unitary. Define $T_y^\gamma : \mathfrak{H}_{T^\gamma y} \rightarrow \mathfrak{H}_y$ to be the zero map when $y \notin Y_0$. The family of maps $\{T_y^\gamma : y \in Y\}$ induces a map T^γ on sections of $Y * \mathfrak{H}$ such that $(T^\gamma f)_y = T_y^\gamma f_{T^\gamma y}$. If f is a measurable section then so is $T^\gamma f$. Also $T^\eta(T^\gamma f) = T^{\gamma\eta} f$ for all γ, η in Γ .

It remains to relate the Borel Hilbert bundle associated with a relatively independent joining $\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2 \rightarrow \mathbf{Y}$ to the Hilbert bundles associated with the extensions $\mathbf{X}_1 \rightarrow \mathbf{Y}$ and $\mathbf{X}_2 \rightarrow \mathbf{Y}$. Let \mathcal{A}_1 and \mathcal{A}_2 be countable, Γ -invariant algebras that generate \mathcal{B}_1 and \mathcal{B}_2 respectively. The countable algebra generated by $\mathcal{A}_1 \otimes \mathcal{A}_2$ is Γ -invariant and generates $\mathcal{B}_1 \otimes \mathcal{B}_2$. We can thus simultaneously form the Borel Hilbert bundles $Y * \mathfrak{H}_1$, $Y * \mathfrak{H}_2$ and $Y * \mathfrak{H}$ corresponding to the extensions $\mathbf{X}_1 \rightarrow \mathbf{Y}$, $\mathbf{X}_2 \rightarrow \mathbf{Y}$ and $\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2 \rightarrow \mathbf{Y}$ respectively. From (2.4) we see that $\mathfrak{H}_y = \mathfrak{H}_{1,y} \otimes \mathfrak{H}_{2,y}$ for μ almost every y . Thus for any section H of $Y * \mathfrak{H}$ and almost every y the corresponding member H_y of \mathfrak{H}_y induces a compact operator $H_y : \mathfrak{H}_{1,y} \rightarrow \mathfrak{H}_{2,y}$ defined by

$$(\phi \star H_y)(x_2) = \int \phi_1(x_1) \cdot H(x_1, x_2) d\mu_{1,y}(x_1) \quad (3.1)$$

for any $\phi \in \mathfrak{H}_{1,y}$. This family of operators induces a map taking almost-surely defined sections of $Y * \mathfrak{H}_1$ to almost-surely defined sections of $Y * \mathfrak{H}_2$. If H is a measurable section of $Y * \mathfrak{H}$ the induced map preserves measurability of sections because

$$y \mapsto \langle 1_{A_{1,n}} \star H, 1_{A_{2,m}} \rangle_y = \langle H, 1_{A_{1,n}} \otimes 1_{A_{2,m}} \rangle_y$$

is measurable for all n, m in \mathbb{N} . However, the induced map need not preserve square-integrability. It may happen that f_1 is a square-integrable section of $Y * \mathfrak{H}_1$ and $y \mapsto f_{1,y} \star H_y$ is not a square-integrable section of $Y * \mathfrak{H}_2$. As the following proposition shows, we avoid this problem when the norms $\|H_y\|_y$ are bounded almost-surely

and write $f_1 \star H$ for the element of $L^2(\mathbf{X}_2)$ corresponding to the square integrable section $f_{1,y} \star H_y$ of $Y \star \mathfrak{H}_2$.

Proposition 3.1. *Let H be a section of $Y \star \mathfrak{H}$. If the norms $\|H_y\|_y$ are essentially bounded and f_1 is a square-integrable section of the bundle $Y \star \mathfrak{H}_1$ then $f_{1,y} \star H_y$ is a square-integrable section of $Y \star \mathfrak{H}_2$.*

Proof. See Section F.3 in [Wil07]. \square

A section H of $Y \star \mathfrak{H}$ also defines for almost every y a compact operator $H_y : \mathfrak{H}_{2,y} \rightarrow \mathfrak{H}_{1,y}$ defined by

$$(H_y \star \phi)(x_1) = \int H_y(x_1, x_2) \cdot \phi(x_2) d\mu_{2,y}(x_2)$$

for any $\phi \in \mathfrak{H}_{2,y}$ with similar properties.

Given a measurable section H of $Y \star \mathfrak{H}$ we can spectrally decompose the compact operator $H_y : \mathfrak{H}_{1,y} \rightarrow \mathfrak{H}_{2,y}$ for almost every $y \in Y$. The following theorem, due to Furstenberg and Katznelson, shows that when $\mathbf{X}_1 = \mathbf{X}_2$ and H is positive-definite and symmetric, the spectral decomposition is measurable.

Theorem 3.2 (3.7 in [FK91]). *Let $\mathbf{X} \rightarrow \mathbf{Y}$ be an extension of systems. Form the corresponding Borel Hilbert bundle $Y \star \mathfrak{H}$. Let H_y be a measurable family of positive-definite, self-adjoint, compact operators on \mathfrak{H}_y . Let $\lambda_n(y)$ be a decreasing enumeration of the positive eigenvalues of H_y , counting multiplicities. There is a sequence Ψ_n of square integrable sections of $Y \star \mathfrak{H}$ such that $\Psi_{n,y} \star H_y = \lambda_n(y) \Psi_{n,y}$ whenever $\lambda_n(y)$ is defined, $\Psi_{n,y} = 0$ otherwise, and $\{\Psi_{n,y} : n \in \mathbb{N}\} \setminus \{0\}$ is orthonormal in almost every fiber.*

4. ALMOST-PERIODIC FUNCTIONS AND EIGENFUNCTIONS

We will describe the characteristic factors $\mathcal{C}_{k,i}$ in terms of almost periodic functions. In this section we prove the results about almost-periodic functions and eigenfunctions that we will need later. Most of the results in this section are well-known in one form or another; we provide the details for the sake of completion.

Let $\mathbf{X} \rightarrow \mathbf{Y}$ be an extension and let μ_y be an almost-surely defined disintegration of μ over \mathbf{Y} . We say that f in $L^2(\mathbf{X})$ is *almost-periodic* for this extension if for every $\varepsilon > 0$ one can find a finite subset Ξ of $L^\infty(\mathbf{X})$ and $E \subset Y$ with $\mu(E) > 1 - \varepsilon$ such that

$$\min\{\|T^\gamma f - \xi\|_y : \xi \in \Xi\} \leq \varepsilon \quad (4.1)$$

for each $\gamma \in \Gamma$ and almost every $y \in E$. The closure of the set of almost-periodic functions, which we denote $\mathcal{A}(\mathbf{X}|\mathbf{Y})$, forms a closed subspace of $L^2(\mathbf{X})$ that contains the constant functions. Also, if f is almost-periodic then so is $|f|$. Thus condition (c) in Lemma 3.1 of [FK91] is satisfied and there exists a sub- σ -algebra \mathcal{C} of \mathcal{B} such that $\mathcal{A}(\mathbf{X}|\mathbf{Y}) = L^2(X, \mathcal{C}, \mu)$. This lets us approximate any f in $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ arbitrarily well by a function in $L^\infty(X, \mathcal{B}, \mu)$ that is almost-periodic over \mathbf{Y} as follows: truncate f at a high level and then re-define f to be zero on certain fibers of the factor map as in the proof of Theorem 9.1 in [FKO82]. Since $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ is closed and invariant under T^γ for each $\gamma \in \Gamma$, it is also T invariant. Thus \mathcal{C} is T invariant. When \mathbf{Y} is the trivial factor, write $\mathcal{A}(\mathbf{X})$ for $\mathcal{A}(\mathbf{X}|\mathbf{Y})$.

We say that an extension $\mathbf{X} \rightarrow \mathbf{Y}$ is *compact* if $\mathcal{A}(\mathbf{X}|\mathbf{Y}) = L^2(\mathbf{X})$ and *weak-mixing* if $\mathcal{A}(\mathbf{X}|\mathbf{Y}) = L^2(\mathbf{Y})$. Given sub- σ -algebras \mathcal{D} and \mathcal{E} of \mathcal{B} , we will say that

$\mathcal{D} \rightarrow \mathcal{E}$ is compact for T if \mathcal{D} and \mathcal{E} are T invariant sub- σ -algebras of \mathcal{B} and the corresponding extension $\mathbf{Y} \rightarrow \mathbf{Z}$ is compact.

Fix systems \mathbf{X}_1 and \mathbf{X}_2 having a common factor \mathbf{Y} . Let \mathcal{C}_1 and \mathcal{C}_2 be the σ -algebras corresponding to $\mathcal{A}(\mathbf{X}_1|\mathbf{Y})$ and $\mathcal{A}(\mathbf{X}_2|\mathbf{Y})$ respectively. We begin by relating $\mathcal{A}(\mathbf{X}_1|\mathbf{Y})$ and $\mathcal{A}(\mathbf{X}_2|\mathbf{Y})$ to $\mathcal{A}(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2|\mathbf{Y})$ by showing that any H in $L^\infty(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ that is almost-periodic over \mathbf{Y} satisfies

$$\langle H, f_1 \otimes f_2 \rangle = \langle H, \mathbb{E}(f_1|\mathcal{C}_1) \otimes \mathbb{E}(f_2|\mathcal{C}_2) \rangle \quad (4.2)$$

for any f_1 in $L^\infty(\mathbf{X}_1)$ and any f_2 in $L^\infty(\mathbf{X}_2)$. This is similar to Proposition 4.4.4 in [McC99].

Proposition 4.1. *For any H in $L^\infty(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ almost-periodic over \mathbf{Y} and any f_1 in $L^2(\mathbf{X}_1)$ the element $f_1 \star H$ of $L^2(\mathbf{X}_2)$ is almost-periodic over \mathbf{Y} .*

Proof. It suffices to prove this when f is in $L^\infty(\mathbf{X}_1)$. Fix $\varepsilon > 0$. We have to find a finite subset Ξ of $L^\infty(\mathbf{X}_2)$ and a subset E of Y with $\mu(E) > 1 - \varepsilon$ such that

$$\min\{\|T_2^\gamma(f_1 \star H) - \xi\|_y : \xi \in \Xi\} \leq \varepsilon \quad (4.3)$$

for each $\gamma \in \Gamma$ and almost every $y \in E$. Almost-periodicity of H over \mathbf{Y} implies the existence of a finite subset Ψ of $L^\infty(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ and a subset F_1 of Y with $\nu(F_1) > 1 - \varepsilon/16$ such that

$$\min\{\|(T_1^\gamma \times T_2^\gamma)H - \psi\|_y : \psi \in \Psi\} < \varepsilon/16$$

for all $\gamma \in \Gamma$ and almost all $y \in F_1$. Write $\Psi = \{\psi_1, \dots, \psi_k\}$. Let γ_n be an enumeration of Γ . For each $1 \leq i \leq k$ and almost every y we have a compact operator $\psi_{i,y}$ mapping $\mathfrak{H}_{1,y}$ to $\mathfrak{H}_{2,y}$. Thus for each $1 \leq i \leq k$ and almost every y we can find a positive integer $M_i(y)$ such that

$$\{(T_1^{\gamma_n} f_{1,T^{\gamma_n}y}) \star \psi_{i,y} : 1 \leq n \leq M_i(y)\}$$

is $\varepsilon/16$ -dense in $\{(T_1^\gamma f_{1,T^\gamma y}) \star \psi_{i,y} : \gamma \in \Gamma\}$. Each of the functions M_i is measurable. Thus we can find N in \mathbb{N} so large that $F_2 = M_1^{-1}[1, N] \cap \dots \cap M_k^{-1}[1, N]$ has measure at least $1 - \varepsilon/16$. Put

$$\Xi = \{(T^{\gamma_n} f_1) \star \psi_i : 1 \leq i \leq k, 1 \leq n \leq N\}$$

and $E = F_1 \cap F_2$. Fix γ in Γ and almost any y in E . We can choose i such that

$$\|(T_1^\gamma \times T_2^\gamma)H - \psi_i\|_y \leq \varepsilon/16$$

and then guarantee

$$\|(T_1^\gamma f_{1,T^\gamma y}) \star \psi_{i,y} - (T_1^{\gamma_n} f_{1,T^{\gamma_n}y}) \star \psi_{i,y}\|_y \leq \varepsilon/16$$

holds for some $1 \leq n \leq N$. From

$$\begin{aligned} (T_2^\gamma(f_1 \star H))_y(x_2) &= \int f_1(x_1) H(x_1, T_2^\gamma x_2) d\mu_{1,T^\gamma y}(x_1) \\ &= \int f_1(T_1^\gamma x_1) H(T_1^\gamma x_1, T_2^\gamma x_2) d\mu_{1,y}(x_1) \end{aligned}$$

we have

$$\begin{aligned} &\|T_2^\gamma(f_1 \star H) - (T_1^{\gamma_n} f_1) \star \psi_i\|_y \\ &\leq \|T_2^\gamma(f_1 \star H) - (T_1^\gamma f_1) \star \psi_i\|_y + \|(T_1^\gamma f_1) \star \psi_i - (T_1^{\gamma_n} f_1) \star \psi_i\|_y \\ &\leq \|(T_1^\gamma f_1) \star ((T_1^\gamma \times T_2^\gamma)H) - (T_1^\gamma f_1) \star \psi_i\|_y + \|(T_1^\gamma f_1) \star \psi_i - (T_1^{\gamma_n} f_1) \star \psi_i\|_y \end{aligned}$$

so (4.3) holds as desired. \square

Proposition 4.2. *For any H in $L^\infty(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ almost-periodic over \mathbf{Y} and any f_2 in $L^2(\mathbf{X}_2)$ the element $H \star f_2$ of $L^2(\mathbf{X}_1)$ is almost-periodic over \mathbf{Y} .*

Proof. Identical to the proof of the previous proposition. \square

Proposition 4.3. *For any H in $L^\infty(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ that is almost-periodic over \mathbf{Y} , any f_1 in $L^2(\mathbf{X}_1)$ and any f_2 in $L^2(\mathbf{X}_2)$ we have (4.2).*

Proof. We have

$$\langle H, f_1 \otimes f_2 \rangle = \langle H, (f_1 - \mathbb{E}(f_1|\mathcal{C}_1) + \mathbb{E}(f_1|\mathcal{C}_1)) \otimes f_2 \rangle$$

and a similar equality holds for f_2 so it suffices to prove that $\langle H, f_1 \otimes f_2 \rangle$ is zero when either f_1 is orthogonal to \mathcal{C}_1 or f_2 is orthogonal to \mathcal{C}_2 . The two cases are similar. In the latter we have

$$\begin{aligned} \langle H, f_1 \otimes f_2 \rangle &= \iint f_1(x_1) H(x_1, x_2) f_2(x_2) d(\mu_{1,y} \otimes \mu_{2,y})(x_1, x_2) d\mu(y) \\ &= \iint (f_1 \star H)(x_2) f_2(x_2) d\mu_{2,y}(x_2) d\mu(y) = \langle f_1 \star H, f_2 \rangle \end{aligned}$$

which is zero by Proposition 4.1. \square

Corollary 4.4. *Let \mathcal{I} be the sub- σ -algebra of $T_1 \times T_2$ invariant sets in $\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2$. If f_1 in $L^\infty(\mathbf{X}_1)$ is orthogonal to $\mathcal{A}(\mathbf{X}_1|\mathbf{Y})$ or f_2 in $L^\infty(\mathbf{X}_2)$ is orthogonal to $\mathcal{A}(\mathbf{X}_2|\mathbf{Y})$ then $\mathbb{E}(f_1 \otimes f_2|\mathcal{I}) = 0$ in $L^2(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$.*

Proof. For any $T_1 \times T_2$ invariant function H in $L^2(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ we have

$$\langle \mathbb{E}(f_1 \otimes f_2|\mathcal{I}), H \rangle = \langle f_1 \otimes f_2, H \rangle = \langle \mathbb{E}(f_1|\mathcal{C}_1) \otimes \mathbb{E}(f_2|\mathcal{C}_2), H \rangle = 0$$

by (4.2), because invariant functions are certainly almost-periodic. \square

Theorem 4.5. $\mathcal{A}(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2|\mathbf{Y}) = \mathcal{A}(\mathbf{X}_1|\mathbf{Y}) \otimes \mathcal{A}(\mathbf{X}_2|\mathbf{Y})$.

Proof. It is straightforward to check that if f_1 in $L^\infty(\mathbf{X}_1)$ and f_2 in $L^\infty(\mathbf{X}_2)$ are both almost-periodic over \mathbf{Y} then $f_1 \otimes f_2$ in $L^2(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2)$ is almost-periodic over \mathbf{Y} . On the other hand, if H belongs to $\mathcal{A}(\mathbf{X}_1 \times_{\mathbf{Y}} \mathbf{X}_2|\mathbf{Y})$ and is orthogonal to $\mathcal{A}(\mathbf{X}_1|\mathbf{Y}) \otimes \mathcal{A}(\mathbf{X}_2|\mathbf{Y})$ then by Proposition 4.3 we have $\langle H, f_1 \otimes f_2 \rangle = 0$ for all f_1 in $L^\infty(\mathbf{X}_1)$ and all f_2 in $L^\infty(\mathbf{X}_2)$ so $H = 0$. \square

Recall that a function f in $L^2(\mathbf{X})$ is *weakly mixing* for $\mathbf{X} \rightarrow \mathbf{Y}$ if

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} |\mathbb{E}(\phi \cdot T^g f|\mathbf{Y})|^2 d\mu dm(g) = 0$$

for every ϕ in $L^\infty(\mathbf{X})$. The set $\mathcal{W}(\mathbf{X}|\mathbf{Y})$ of weakly mixing functions is a closed, T invariant subspace of $L^2(\mathbf{X})$. Proposition 4.1 lets us prove the following result.

Theorem 4.6. *For any extension $\mathbf{X} \rightarrow \mathbf{Y}$ we have $L^2(\mathbf{X}) = \mathcal{A}(\mathbf{X}|\mathbf{Y}) \oplus \mathcal{W}(\mathbf{X}|\mathbf{Y})$.*

Proof. First we show that if f in $L^2(\mathbf{X})$ is orthogonal to $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ then f belongs to $\mathcal{W}(\mathbf{X}|\mathbf{Y})$. Fix ϕ in $L^\infty(\mathbf{X})$. Let \mathcal{I} denote the sub- σ -algebra of $T \times T$ invariant

sets and put $H = \mathbb{E}(\phi \otimes \phi | \mathcal{I})$ in $L^2(\mathbf{X} \times_{\mathbf{Y}} \mathbf{X})$. We have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} |\mathbb{E}(\phi \cdot T^g f | \mathbf{Y})|^2 d\mu dm(g) \\ &= \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} (\phi \otimes \phi) \cdot (T \times T)^g(f \otimes f) d\nu dm(g) \\ &= \int \mathbb{E}(\phi \otimes \phi | \mathcal{I}) \cdot (f \otimes f) d\nu = \langle H, f \otimes f \rangle = \langle f \star H, f \rangle = 0 \end{aligned}$$

by the mean ergodic theorem and Proposition 4.1. Since ϕ was arbitrary, f is weakly mixing over \mathbf{Y} .

Now we show that $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ and $\mathcal{W}(\mathbf{X}|\mathbf{Y})$ are orthogonal. Fix f in $\mathcal{W}(\mathbf{X}|\mathbf{Y})$. It suffices to prove that f is orthogonal to any ϕ in $L^\infty(\mathbf{X})$ that is almost-periodic over \mathbf{Y} . Fix $\varepsilon > 0$. Since ϕ is almost periodic we can find a subset E of Y with $\mu(E) > 1 - \varepsilon$ and a finite subset $\Xi = \{\xi_1, \dots, \xi_k\}$ of $L^2(\mathbf{X})$ such that (4.1) holds for all $\gamma \in \Gamma$ and all $y \in E$. Fix $g \in G$. Since Γ is dense in G we can find some γ in Γ such that $\|T^\gamma \phi - T^g \phi\|_y < \varepsilon$ for all y in a subset E_g of Y with $\mu(E_g) > 1 - \varepsilon$. For each y in E choose $1 \leq \iota(y) \leq k$ so that $\|T^\gamma \phi - \xi_{\iota(y)}\|_y < \varepsilon$. Put $F = E \cap E_g$. Cauchy-Schwarz gives

$$\begin{aligned} \int T^g \phi \cdot T^g f d\mu_y &\leq \left| \int \xi_{\iota(y)} \cdot T^g f d\mu_y \right| + 2\varepsilon \|T^g f\|_y \\ &\leq \sum_{i=1}^k |\mathbb{E}(\xi_i \cdot T^g f | \mathbf{Y})(y)| + 2\varepsilon \|T^g f\|_y \end{aligned} \tag{4.4}$$

for any $y \in F$. Combining this with

$$\int T^g \phi \cdot T^g f d\mu_y \leq \int |T^g f| d\mu_y \|\phi\|_\infty \leq \|T^g f\|_y \|\phi\|_\infty \tag{4.5}$$

which holds (in particular) for almost-every $y \notin F$ we get

$$\begin{aligned} |\langle \phi, f \rangle| &\leq \sum_{i=1}^k \int |\mathbb{E}(\xi_i \cdot T^g f | \mathbf{Y})| d\mu + 2\varepsilon \|T^g f\| + \int 1_{Y \setminus F}(y) \cdot \|T^g f\|_y d\mu(y) \|\phi\|_\infty \\ &\leq \sum_{i=1}^k \int |\mathbb{E}(\xi_i \cdot T^g f | \mathbf{Y})| d\mu + 2\varepsilon \|f\| + \sqrt{2\varepsilon} \cdot \|f\| \cdot \|\phi\|_\infty \end{aligned}$$

by integrating, applying Cauchy-Schwarz, and noting that $\mu(Y \setminus F) \leq 2\varepsilon$. Finally, averaging over the Følner sequence Φ and applying Cauchy-Schwarz once more gives

$$|\langle \phi, f \rangle| \leq 2\varepsilon \|f\| + \sqrt{2\varepsilon} \|f\| \|\phi\|_\infty + \sum_{i=1}^k \left(\frac{1}{m(\Phi_N)} \int \int_{\Phi_N} |\mathbb{E}(\xi_i \cdot T^g f | \mathbf{Y})|^2 d\mu dm(g) \right)^{1/2}$$

which, upon using the fact that f is weakly-mixing and noting that ε was arbitrary, gives $\langle \phi, f \rangle = 0$. \square

Since the definition of $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ is independent of the Følner sequence Φ , the above proposition implies that $\mathcal{W}(\mathbf{X}|\mathbf{Y})$ is also independent of Φ .

We will use Theorem 4.6 to relate $\mathcal{A}(\mathbf{X}|\mathbf{Y})$ to the eigenfunctions of an extension. Given an extension $\mathbf{X} \rightarrow \mathbf{Y}$, the factor map lets us embed $L^\infty(\mathbf{Y})$ in $L^\infty(\mathbf{X})$.

Thus we can think of $L^2(\mathbf{X})$ as an $L^\infty(\mathbf{Y})$ module. A function f in $L^2(\mathbf{X})$ is an *eigenfunction* over \mathbf{Y} if the closed subspace \mathcal{M} spanned by the orbit of f is a finite-rank $L^\infty(\mathbf{Y})$ module. This means we can find ϕ_1, \dots, ϕ_d in $L^2(\mathbf{X})$ such that

$$\{\alpha^1 \phi_1 + \dots + \alpha^d \phi_d : \alpha^1, \dots, \alpha^d \in L^\infty(\mathbf{Y})\}$$

is dense in \mathcal{M} . Denote by $\mathcal{E}(\mathbf{X}|\mathbf{Y})$ the closed subspace of $L^2(\mathbf{X})$ spanned by the eigenfunctions over \mathbf{Y} . When \mathbf{Y} is the trivial factor, write $\mathcal{E}(\mathbf{X})$ for $\mathcal{E}(\mathbf{X}|\mathbf{Y})$.

Theorem 4.7. *For any extension $\mathbf{X} \rightarrow \mathbf{Y}$ we have $L^2(\mathbf{X}) = \mathcal{E}(\mathbf{X}|\mathbf{Y}) \oplus \mathcal{W}(\mathbf{X}|\mathbf{Y})$.*

Proof. Using the fact that the orbit of an eigenfunction is contained in a finite-rank $L^\infty(\mathbf{Y})$ -module, one can show that every eigenfunction over \mathbf{Y} is almost-periodic over \mathbf{Y} . Thus $\mathcal{E}(\mathbf{X}|\mathbf{Y}) \subset \mathcal{W}(\mathbf{X}|\mathbf{Y})^\perp$ by Theorem 4.6.

It remains to prove that $\mathcal{E}(\mathbf{X}|\mathbf{Y})^\perp \subset \mathcal{W}(\mathbf{X}|\mathbf{Y})$. Fix f in $L^2(\mathbf{X})$ orthogonal to $\mathcal{E}(\mathbf{X}|\mathbf{Y})$. For any ϕ in $L^\infty(\mathbf{X})$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} |\mathbb{E}(\phi \cdot T^g f | \mathbf{Y})|^2 d\mu dm(g) = \langle f \star H, f \rangle$$

as in the proof of Theorem 4.6, where $H = \mathbb{E}(\phi \otimes \phi | \mathcal{I})$. Thus it suffices to prove that $f \star H$ is in $\mathcal{E}(\mathbf{X}|\mathbf{Y})$. Let Ψ_n and λ_n be as in Theorem 3.2. Since $\{\Psi_{n,y} : n \in \mathbb{N}\}$ spans the image of H_y for almost-every y , it suffices to prove that each Ψ_n is in $\mathcal{E}(\mathbf{X}|\mathbf{Y})$. To this end, fix n in \mathbb{N} and denote by $\theta(y)$ the multiplicity of the eigenvalue $\lambda_n(y)$ if $\lambda_n(y)$ is defined, and put $\theta(y) = 0$ otherwise. Each of the functions λ_m is measurable, so θ is too. For each $k \in \mathbb{N}$ let $\Omega_k = \theta^{-1}(k)$.

We will show that the orbit of $1_{\Omega_k}(y)\Psi_n$ is a finite-rank $L^\infty(\mathbf{Y})$ module. Fix γ in Γ . We have $T^\gamma H_{T^\gamma y} = H_y T^\gamma$ for almost-every y because H is $T \times T$ invariant. Since T^γ is unitary on almost every fiber, the operators H_y and $H_{T^\gamma y}$ have the same spectrum, so each of the functions λ_m is T^γ invariant. This implies Ω_k is T^γ -invariant. Also

$$(H \star T^\gamma \Phi_n)_y = (T^\gamma(H \star \Phi_n))_y = T^\gamma(H_{T^\gamma y} \star \Phi_{n,T^\gamma y}) = \lambda_n(y)(T^\gamma \Psi_n)_y$$

so in almost every fiber, the dimension the Γ orbit of the square-integrable section $y \mapsto 1_{\Omega_k}(y)\Psi_{n,y}$ of $Y \star \mathfrak{H}$ is bounded by k . Thus $y \mapsto 1_{\Omega_k}(y)\Psi_{n,y}$ corresponds to an eigenfunction. Summing over k proves that Ψ_n is in $\mathcal{E}(\mathbf{X}|\mathbf{Y})$ as desired. \square

Combining Theorems 4.6 and 4.7 yields the following result, a basic version of which will be used later.

Corollary 4.8. *For any extension $\mathbf{X} \rightarrow \mathbf{Y}$ we have $\mathcal{A}(\mathbf{X}|\mathbf{Y}) = \mathcal{E}(\mathbf{X}|\mathbf{Y})$.*

5. THE CHARACTERISTIC FACTORS

In this section we define the characteristic factors $\mathcal{C}_{k,i}$ associated to commuting, measurable actions T_1, \dots, T_k and prove that

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \prod_{i=1}^k T_k^g \dots T_i^g f_i - \prod_{i=1}^k T_k^g \dots T_i^g \mathbb{E}(f_i | \mathcal{C}_{k,i}) dm(g) = 0 \quad (5.1)$$

in $L^2(X, \mathcal{B}, \mu)$ for any f_1, \dots, f_k in $L^\infty(X, \mathcal{B}, \mu)$.

To define the $\mathcal{C}_{k,i}$ fix k in \mathbb{N} and let T_1, \dots, T_k be commuting, measurable actions of G on a probability space (X, \mathcal{B}, μ) . Let $\mathcal{C}_{1,1}$ be the sub- σ -algebra of T_1 invariant sets. It is invariant under all of the actions T_2, \dots, T_k because they each commute

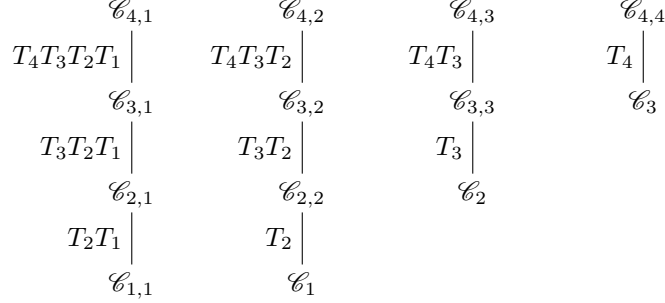


FIGURE 1. The sub- σ -algebras $\mathcal{C}_{k,i}$ for $k \leq 4$. A line indicates that the upper σ -algebra corresponds to the functions almost-periodic for the labeled action over the lower σ -algebra.

with T_1 . Suppose by induction that for some $1 \leq l \leq k-1$ we have defined sub- σ -algebras $\mathcal{C}_{l,1}, \dots, \mathcal{C}_{l,l}$ such that

- (1) for each $1 \leq j \leq l$, $\mathcal{C}_{l,j}$ is $T_l \cdots T_j$ invariant;
- (2) for each $1 \leq j \leq l$ and every $l+1 \leq i \leq k$, $\mathcal{C}_{l,j}$ is T_i invariant.

For each $1 \leq j \leq l$, let \mathbf{Y}_j be the factor of $\mathbf{X}_j = (X, \mathcal{B}, \mu, T_{l+1} \cdots T_j)$ corresponding to $\mathcal{C}_{l,j}$ and let $\mathcal{C}_{l+1,j}$ be the sub- σ -algebra of \mathcal{B} corresponding to $\mathcal{A}(\mathbf{X}_j | \mathbf{Y}_j)$. It is invariant under $T_{l+1} \cdots T_j$ because it consists of $T_{l+1} \cdots T_j$ almost periodic functions, and (if $l < k-1$) it is T_i invariant for all $l+2 \leq i \leq k$ because the actions commute. Let \mathbf{Y}_{l+1} be the factor of $\mathbf{X}_{l+1} = (X, \mathcal{B}, \mu, T_{l+1})$ corresponding to $\mathcal{C}_{l,1} \vee \cdots \vee \mathcal{C}_{l,l}$ and let $\mathcal{C}_{l+1,l+1}$ be the sub- σ -algebra corresponding to $\mathcal{A}(\mathbf{X}_{l+1} | \mathbf{Y}_{l+1})$. It is T_{l+1} invariant because it consists of the T_{l+1} almost-periodic functions over \mathbf{Y}_{l+1} , and (if $l < k-1$) it is T_i invariant for all $l+2 \leq i \leq k$ because the actions commute. This concludes the inductive construction. Figure 1 shows how the $\mathcal{C}_{k,i}$ are related for $k \leq 4$. The remainder of this section constitutes a proof of the following theorem.

Theorem 5.1. *Let T_1, \dots, T_k be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Then*

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \prod_{i=1}^k T_k^g \cdots T_i^g f_i - \prod_{i=1}^k T_k^g \cdots T_i^g \mathbb{E}(f_i | \mathcal{C}_{k,i}) \, dm(g) = 0$$

for any f_i in $L^\infty(X, \mathcal{B}, \mu)$.

Since the limit

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \prod_{i=1}^k T_k^g \cdots T_i^g f_i \, dm(g)$$

is known to exist (see [ZK11]) it suffices to prove that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \int f_{k+1} \cdot \prod_{i=1}^k T_k^g \cdots T_i^g f_i \, d\mu \, dm(g) \\ &= \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \int f_{k+1} \cdot \prod_{i=1}^k T_k^g \cdots T_i^g \mathbb{E}(f_i | \mathcal{C}_{k,i}) \, d\mu \, dm(g) \end{aligned} \tag{5.2}$$

for any f_1, \dots, f_{k+1} in $L^\infty(X, \mathcal{B}, \mu)$. We will prove (5.2) by induction on k . The case $k = 1$ follows from the mean ergodic theorem: we have

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} T_1^g f_1 \cdot f_2 \, d\mu \, dm(g) = \int \mathbb{E}(f_1 | \mathcal{C}_{1,1}) \cdot f_2 \, d\mu$$

by Proposition 2.2, which can be re-written as

$$\int f_1 \otimes f_2 \, d\nu_1 = \int \mathbb{E}(f_1 | \mathcal{C}_{1,1}) \otimes f_2 \, d\nu_1$$

where ν_1 is the Furstenberg joining for the action T_1 . For the inductive step we need the following application of the van der Corput trick, which is a version of Lemma 4.7 in [Aus10].

Theorem 5.2. *Let T_1, \dots, T_k be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Let ν_{k-1} be the Furstenberg joining of the actions T_1, \dots, T_{k-1} and let ν_k be the Furstenberg joining of the actions T_1, \dots, T_k . Suppose we have sub- σ -algebras $\mathcal{E}_1, \dots, \mathcal{E}_{k-1}$ with each \mathcal{E}_i invariant under $T_{k-1} \cdots T_i$ and T_k such that*

$$\int f_1 \otimes \cdots \otimes f_k \, d\nu_{k-1} = \int \mathbb{E}(f_1 | \mathcal{E}_1) \otimes \cdots \otimes \mathbb{E}(f_{k-1} | \mathcal{E}_{k-1}) \otimes f_k \, d\nu_{k-1}$$

for all f_1, \dots, f_k in $L^\infty(X, \mathcal{B}, \mu)$. Put $\mathcal{E}_k = \mathcal{E}_1 \vee \cdots \vee \mathcal{E}_{k-1}$. Then

$$\int f_1 \otimes \cdots \otimes f_{k+1} \, d\nu_k = \int \mathbb{E}(f_1 | \mathcal{F}_1) \otimes \cdots \otimes \mathbb{E}(f_k | \mathcal{F}_k) \otimes f_{k+1} \, d\nu_k \quad (5.3)$$

for all f_1, \dots, f_{k+1} in $L^\infty(X, \mathcal{B}, \mu)$ where, for each $1 \leq i \leq k$ the sub- σ -algebra \mathcal{F}_i corresponds to the functions that are $T_k \cdots T_i$ almost-periodic over \mathcal{E}_i .

Proof. Fix f_1, \dots, f_{k+1} in $L^\infty(X, \mathcal{B}, \mu)$ with $\|f_i\|_\infty \leq 1$ for all $1 \leq i \leq k+1$. Since \mathcal{E}_i is $T_{k-1} \cdots T_i$ invariant and contained in \mathcal{E}_k for each $1 \leq i \leq k-1$, we have

$$T_{k-1}^g \cdots T_i^g \mathbb{E}(f_i | \mathcal{E}_i) = \mathbb{E}(T_{k-1}^g \cdots T_i^g \mathbb{E}(f_i | \mathcal{E}_i) | \mathcal{E}_k)$$

for each $1 \leq i \leq k-1$. Thus we can re-write our assumption as

$$\int f_1 \otimes \cdots \otimes f_k \, d\nu_{k-1} = \int \mathbb{E}(f_1 | \mathcal{E}_1) \otimes \cdots \otimes \mathbb{E}(f_k | \mathcal{E}_k) \, d\nu_{k-1} \quad (5.4)$$

using (2.3). We proceed by applying the van der Corput trick to the sequence

$$u(g) = \prod_{i=1}^k T_k^g \cdots T_i^g f_i$$

in $L^2(X, \mathcal{B}, \mu)$. From (2.3) and (5.4) we see that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \langle u(hg), u(lg) \rangle \, dm(g) &= \int \bigotimes_{i=1}^k (T_k^h \cdots T_i^h f_i \cdot T_k^l \cdots T_i^l f_i) \, d\nu_{k-1} \\ &= \int \bigotimes_{i=1}^k \mathbb{E}(T_k^h \cdots T_i^h f_i \cdot T_k^l \cdots T_i^l f_i | \mathcal{E}_i) \, d\nu_{k-1} \end{aligned}$$

for any $h, l \in G$. Using (2.3) once more yields

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \langle u(hg), u(lg) \rangle \, dm(g) \leq \|\mathbb{E}(T_k^h \cdots T_i^h f_i \cdot T_k^l \cdots T_i^l f_i | \mathcal{E}_i)\|$$

for each $1 \leq i \leq k$, the norm taken in $L^2(X, \mathcal{B}, \mu)$. Let \mathbf{X}_i be the system $(X, \mathcal{B}, \mu, T_k \cdots T_i)$ and let \mathbf{Y}_i be a factor corresponding to \mathcal{E}_i . Let \mathcal{I}_i be the sub- σ -algebra of $T_k \cdots T_i \times T_k \cdots T_i$ invariant sets in the relatively independent joining $\mathbf{X}_i \times_{\mathbf{Y}_i} \mathbf{X}_i$. We have

$$\begin{aligned} & \limsup_{H \rightarrow \infty} \frac{1}{m(\Phi_H)^2} \int \int_{\Phi_H \Phi_H} \|\mathbb{E}(T_k^h \cdots T_i^h f_i \cdot T_k^l \cdots T_i^l f_i | \mathcal{E}_i)\| \, dm(h) \, dm(l) \\ & \leq \lim_{H \rightarrow \infty} \left\| \frac{1}{m(\Phi_H)} \int (T_k \cdots T_i \times T_k \cdots T_i)^h (f_i \otimes f_i) \, dm(h) \right\| = \|\mathbb{E}(f_i \otimes f_i | \mathcal{I}_i)\| \end{aligned}$$

in $L^2(\mathbf{X}_i \times_{\mathbf{Y}_i} \mathbf{X}_i)$ by Cauchy-Schwarz and the mean ergodic theorem. By Corollary 4.4 the conditional expectation $\mathbb{E}(f_i \otimes f_i | \mathcal{I}_i)$ will be zero if f_i is orthogonal to $\mathcal{A}(\mathbf{X}_i | \mathbf{Y}_i)$. Since $1 \leq i \leq k$ was arbitrary, (5.3) follows from the van der Corput trick. \square

Taking $\mathcal{E}_i = \mathcal{C}_{k-1, i}$ in the preceding theorem proves (5.2) and concludes the proof of Theorem 5.1. We conclude this section with another application of the van der Corput trick that is sometimes useful.

Theorem 5.3. *Let T_1, \dots, T_k be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Let ν_{k-1} be the Furstenberg joining of the actions T_1, \dots, T_{k-1} and let ν_k be the Furstenberg joining of the actions T_1, \dots, T_k . Let \mathcal{I}_k denote the sub- σ -algebra of \mathcal{B}^k consisting of*

$$T_k T_{k-1} \cdots T_1 \times \cdots \times T_k T_{k-1} \times T_k$$

invariant sets. If $\mathbb{E}(f_1 \otimes \cdots \otimes f_k | \mathcal{I}_k) = 0$ in $L^2(X^k, \mathcal{B}^k, \nu_{k-1})$ for some f_1, \dots, f_k in $L^\infty(X, \mathcal{B}, \mu)$ then

$$\int f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} \, d\nu_k = 0$$

for all f_{k+1} in $L^2(X, \mathcal{B}, \mu)$.

Proof. Fix f_1, \dots, f_k in $L^\infty(X, \mathcal{B}, \mu)$ satisfying $\mathbb{E}(f_1 \otimes \cdots \otimes f_k | \mathcal{I}_k) = 0$. Applying the van der Corput trick as in Theorem 5.2 gives

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \langle u(hg), u(lg) \rangle \, dm(g) = \int \bigotimes_{i=1}^k (T_k^h \cdots T_i^h f_i \cdot T_k^l \cdots T_i^l f_i) \, d\nu_{k-1}$$

for any $h, l \in G$. From this we get

$$\begin{aligned} & \lim_{H \rightarrow \infty} \frac{1}{m(\Phi_H)^2} \int \int_{\Phi_H \Phi_H} \lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \langle u(hg), u(lg) \rangle \, dm(g) \, dm(h) \, dm(l) \\ & = \lim_{H \rightarrow \infty} \left\| \frac{1}{m(\Phi_H)} \int \bigotimes_{i=1}^k T_k^h \cdots T_i^h f_i \, dm(h) \right\|^2 = \int \mathbb{E}(f_1 \otimes \cdots \otimes f_k | \mathcal{I}_k)^2 \, d\nu_{k-1} \end{aligned}$$

where the norm is determined by ν_{k-1} and the last equality follows from Proposition 2.2. The conclusion follows from the van der Corput trick and the fact that strong convergence implies weak convergence. \square

6. LIFTING POSITIVITY

In this section we prove a technical result, based on Theorem 9.1 in [FKO82], that allows us to lift multiple recurrence from one level of Figure 1 to the next, provided the sub- σ -algebras in the lower level are all equal. In the next section we will use this to prove some multiple recurrence results.

Theorem 6.1. *Let T_1, \dots, T_k be commuting, measurable actions of G on a separated, countably-generated probability space (X, \mathcal{B}, μ) . Let \mathcal{D} be a sub- σ -algebra that is $T_k \cdots T_1$ invariant for all $1 \leq i \leq k$. Suppose that*

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \prod_{i=1}^k T_i^g \mathbb{E}(f | \mathcal{D}) d\mu dm(g) > 0$$

for any $f > 0$ in $L^\infty(X, \mathcal{B}, \mu)$. For each $1 \leq i \leq k$, let \mathcal{E}_i be a sub- σ -algebra of \mathcal{B} that is $T_k \cdots T_i$ invariant, and suppose that $\mathcal{E}_i \rightarrow \mathcal{D}$ is $T_k \cdots T_i$ almost-periodic. Then

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \prod_{i=1}^k T_i^g \mathbb{E}(f | \mathcal{E}_i) d\mu dm(g) > 0$$

for any $f > 0$ in $L^\infty(X, \mathcal{B}, \mu)$.

Proof. It suffices to prove the theorem when $f = 1_B$ for some set $B \in \mathcal{B}$ having positive measure. Let μ_x be the disintegration of μ over \mathcal{D} . For each $1 \leq i \leq k$ write f_i for $\mathbb{E}(1_B | \mathcal{E}_i)$. From

$$\mu(B \cap \{f_i = 0\}) = \int f_i \cdot 1_{\{f_i=0\}} d\mu = 0$$

it follows that f_i is positive on almost all of B . Thus we can find a set D_1 in \mathcal{D} with positive measure and some $\alpha > 0$ such that

$$\int f \cdot f_1 \cdots f_k d\mu_x > \alpha \quad (6.1)$$

for all x in D_1 . Fix $\varepsilon = \alpha/4k$.

For any $x \in X$ and any non-empty subset F of Γ define

$$\mathcal{L}(x, F) = \{(T_k^a \cdots T_1^a f_1, \dots, T_k^a f_k) : a \in F\} \subset L^2(X, \mathcal{B}, \mu_x)^k$$

and equip it with the max norm coming from $\|\cdot\|_x$ on the constituents.

Claim. *There is a subset D_2 of D_1 with positive measure such that $\mathcal{L}(x, \Gamma)$ is totally bounded for each $x \in D_2$.*

Proof. For each j in \mathbb{N} put $\varepsilon_j = \mu(D_1)/2^{j+k+1}$. Since each f_i is $T_k \cdots T_i$ almost-periodic over \mathcal{D} one can find finite subsets Ξ_j^i of $L^\infty(X, \mathcal{B}, \mu)$ and subsets E_j^i with measure at least $1 - \varepsilon_j$ such that for each γ in Γ we have

$$\min\{\|T_k^\gamma \cdots T_i^\gamma f_i - \xi\|_x : \xi \in \Xi_j^i\} < \varepsilon_j$$

for every x in E_j^i . Put

$$D_2 = D_1 \setminus \bigcup_{j=1}^{\infty} E_j^1 \cup \cdots \cup E_j^k$$

and note that $\mu(D_2) \geq \mu(D_1)/2$. \square

We will be interested in separated subsets of $\mathcal{L}(x, \Gamma)$ so define

$$\text{Sep}(F, t) := \bigcap_{a \in F} \bigcap_{b \in F} \bigcup_{i=1}^k \{x \in X : \|T_k^a \cdots T_i^a f_i - T_k^b \cdots T_i^b f_i\|_x > t\}$$

for any finite, non-empty subset F of Γ and any positive t . It belongs to \mathcal{D} and when F is a singleton it is all of X . The fact that $\mathcal{L}(x, \Gamma)$ is totally bounded whenever $x \in D_2$ implies that there is a bound on the cardinality of the finite sets F for which x belongs to $\text{Sep}(F, \varepsilon)$. Thus the \mathcal{D} measurable sets

$$Q(F) := \text{Sep}(F, \varepsilon) \setminus \bigcup \{\text{Sep}(E, \varepsilon) : E \subset \Gamma \text{ with } |F| < |E| < \infty\}$$

cover almost all of D_2 as F runs through the finite subsets of Γ and we can fix a finite, non-empty subset F of Γ such that $Q(F) \cap D_2$ has positive measure. For each x in $Q(F) \cap D_2$ we can find some $n \in \mathbb{N}$ with the property that $x \in \text{Sep}(F, \varepsilon + 1/n)$ because of the strict inequalities and finite number of conditions in the definition of $\text{Sep}(F, \varepsilon)$. Thus we can find some $\eta > 0$ with the property that $Q(F) \cap \text{Sep}(F, \varepsilon + \eta) \cap D_2$ has positive measure. Define a function Ψ by

$$\Psi : Q(F) \cap \text{Sep}(F, \varepsilon + \eta) \cap D_2 \rightarrow [0, 2]^{F \times F \times \{1, \dots, k\}}$$

$$\Psi(x) : (a, b, i) \mapsto \|T_k^a \cdots T_i^a f_i - T_k^b \cdots T_i^b f_i\|_x$$

and partition $[0, 2]^{F \times F \times \{1, \dots, k\}}$ into cubes of side length $\eta/2$. Since Ψ is measurable we can find a cell D in the pull-back partition that has positive measure. Now D belongs to \mathcal{D} , so by hypothesis

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} \int \prod_{i=1}^k T_k^g \cdots T_i^g 1_D \cdot 1_D \, d\mu \, dm(g) > 0$$

and thus there exists $\zeta > 0$ and a subset Δ of G with positive lower density such that

$$\int \prod_{i=1}^k T_k^g \cdots T_i^g 1_D \cdot 1_D \, d\mu > \zeta$$

for any g in Δ .

Claim. For any $g \in \Delta$ there is a subset E_g of $(T_k^g \cdots T_1^g)^{-1} D \cap \cdots \cap (T_k^g)^{-1} D \cap D$ with measure at least $\zeta/2$ such that for any $x \in E_g$ one can find $b \in F$ satisfying $\|T_k^{bg} \cdots T_i^{bg} f_i - f_i\|_x < 2\varepsilon$ for every $1 \leq i \leq k$.

Proof. Fix $g \in \Delta$. Since Γ is dense and each T_i^g is unitary we can find γ in Γ such that

$$\|T_k^{ag} \cdots T_i^{ag} f_i - T_k^{a\gamma} \cdots T_i^{a\gamma} f_i\|^2 \leq \min\{\eta^2 \zeta / 2^{2k+2} |F|, \varepsilon^2\} \quad (6.2)$$

for all $a \in F$ and all $1 \leq i \leq k$. It follows from Chebyshev's inequality that there is a subset E_g of $(T_k^g \cdots T_1^g)^{-1} D \cap \cdots \cap (T_k^g)^{-1} D \cap D$ with $\mu(E_g) \geq \zeta/2$ such that

$$\|T_k^{ag} \cdots T_i^{ag} f_i - T_k^{a\gamma} \cdots T_i^{a\gamma} f_i\|_x \leq \eta/4 \quad (6.3)$$

for all $a \in F$, all $1 \leq i \leq k$ and all $x \in E_g$.

If $1 \in F\gamma$ then the claim follows immediately from (6.2), so assume otherwise. In this case the subset $F\gamma \cup \{1\}$ of the subgroup Γ has cardinality strictly larger than F so x does not belong to $\text{Sep}(F\gamma \cup \{1\}, \varepsilon)$. Thus we can find $\alpha \neq \beta$ in $F\gamma \cup \{1\}$ such that

$$\|T_k^\alpha \cdots T_i^\alpha f_i - T_k^\beta \cdots T_i^\beta f_i\|_x \leq \varepsilon \quad (6.4)$$

for all $1 \leq i \leq k$, and the proof will be concluded if we can show that one of α or β must be 1. Fix $a \neq b$ in F . (If $|F| = 1$ then one of α or β must be 1.) That x belongs to $\text{Sep}(F, \varepsilon + \eta)$ tells us

$$\|T_k^a \cdots T_i^a f_i - T_k^b \cdots T_i^b f_i\|_x > \varepsilon + \eta$$

holds for some $1 \leq i \leq k$. Since $T_k^g \cdots T_i^g x$ belongs to D we must have

$$\|T_k^a \cdots T_i^a f_i - T_k^b \cdots T_i^b f_i\|_{T_k^g \cdots T_i^g x} > \varepsilon + \eta/2$$

because the function $x \mapsto \Psi(x)(a, b, i)$ takes values in an interval of length at most $\eta/2$. Now $a \neq b$ in F were arbitrary so, combined with (6.3), this forces one of α or β to be 1 as otherwise (6.4) is contradicted. \square

We can now finish the proof. Fix $g \in \Delta$ and let E_g be as in the claim. For any $x \in E_g$ we can find some $b \in F$ such that $\|T_k^{bg} \cdots T_i^{bg} f_i - f_i\|_x \leq 2\varepsilon$ for all $1 \leq i \leq k$. Thus

$$\int f \cdot \prod_{i=1}^k T_k^{bg} \cdots T_i^{bg} f_i d\mu_x \geq \alpha - 2k\varepsilon = \frac{\alpha}{2}$$

for any x in the subset E_g of D . Summing over $b \in F$ on the left hand side weakens the inequality and removes the dependence of b on x and g . This allows us to integrate over E_g , obtaining

$$\sum_{b \in F} \int f \cdot \prod_{i=1}^k T_k^{bg} \cdots T_i^{bg} f_i d\mu \geq \frac{\zeta\alpha}{4}$$

which, after averaging over Δ using the Følner sequence Φ , gives

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \prod_{i=1}^k T_k^g \cdots T_i^g f_i d\mu dm(g) \geq \liminf_{N \rightarrow \infty} \frac{m(\Delta \cap \Phi_N)}{m(\Phi_N)} \cdot \frac{\zeta\alpha}{4|F|}$$

concluding the proof. \square

7. RECURRENCE RESULTS

Bergelson's conjecture states that for any commuting actions T_1, \dots, T_k of G on a separated, countably generated probability space (X, \mathcal{B}, μ) we have

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} 1_B \cdot \prod_{i=1}^k T_k \cdots T_i^g 1_B d\mu dm(g) > 0 \quad (7.1)$$

for every B in \mathcal{B} with positive measure. In this section we verify this conjecture when $k = 2$ without additional assumptions, and when $k = 3$ assuming T_1, T_2 and $T_2 T_1$ are ergodic. The $k = 2$ case was previously obtained for countable, amenable groups in [BMZ97].

Theorem 7.1. *Let T_1, T_2 be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Then*

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \cdot T_2^g T_1^g f \cdot T_2^g f d\mu dm(g) > 0 \quad (7.2)$$

for any $f > 0$ in $L^\infty(X, \mathcal{B}, \mu)$.

Proof. By Theorem 5.1 it suffices to prove that

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \cdot T_2^g T_1^g \mathbb{E}(f | \mathcal{C}_{2,1}) \cdot T_2^g \mathbb{E}(f | \mathcal{C}_{2,2}) d\mu dm(g) > 0 \quad (7.3)$$

for all $f > 0$ in $L^\infty(X, \mathcal{B}, \mu)$. If f is of the form 1_B for some $B \in \mathcal{C}_1$ with $\mu(B) > 0$ then $T_2 T_1 f \cdot T_2 f = T_2 f$ so in this case (7.2) follows from the mean ergodic theorem. Thus we have (7.2) whenever f is \mathcal{C}_1 measurable. Applying Theorem 6.1 with $\mathcal{D} = \mathcal{C}_1$, $\mathcal{E}_1 = \mathcal{C}_{2,1}$ and $\mathcal{E}_2 = \mathcal{C}_{2,2}$ yields (7.3). \square

When $k = 3$ we cannot use Theorem 6.1 to prove (7.1) because the sub- σ -algebras $\mathcal{C}_{2,1}$ and $\mathcal{C}_{2,2}$ need not agree and because the behavior of T_3 with respect to the extensions $\mathcal{C}_{2,i} \rightarrow \mathcal{C}_{1,1}$ is unknown. However, if T_1 is ergodic then $\mathcal{C}_{1,1}$ is trivial and the sub- σ -algebras $\mathcal{C}_{2,1}$ and $\mathcal{C}_{2,2}$ consist of functions that are almost-periodic for $T_2 T_1$ and T_2 respectively over the trivial factor. We will prove below that if T_2 and $T_2 T_1$ are ergodic any function almost-periodic for T_2 or $T_2 T_1$ over the trivial factor is necessarily almost periodic for T_3 over the trivial factor. This leads to a description of characteristic factors that allow us, under the aforementioned ergodicity assumptions, to prove Bergelson's conjecture when $k = 3$.

Given a system \mathbf{X} , recall that f in $L^2(\mathbf{X})$ is an *eigenfunction* of \mathbf{X} if its T -orbit is contained in a T -invariant, finite-dimensional subspace of $L^2(\mathbf{X})$. In other words f is an eigenfunction of T if its orbit is contained in a finite-dimensional sub-representation of $L^2(\mathbf{X})$. Denote by $\mathcal{E}(\mathbf{X})$ or $\mathcal{E}(T)$ the closure of the subspace of $L^2(\mathbf{X})$ spanned by the eigenfunctions of \mathbf{X} .

Proposition 7.2. *Let S_1 and S_2 be commuting actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . If S_2 is ergodic then $\mathcal{E}(S_2) \subset \mathcal{E}(S_1)$.*

Proof. Let f be an eigenfunction of S_2 and let \mathcal{M} be an S_2 -invariant, finite-dimensional subspace of $L^2(X, \mathcal{B}, \mu)$ containing the orbit of f . Without loss of generality, we can assume \mathcal{M} is irreducible. Let \mathcal{N} be the closed subspace of $L^2(X, \mathcal{B}, \mu)$ spanned by the sub-representations of G on $L^2(X, \mathcal{B}, \mu)$ induced by S_2 that are equivalent to \mathcal{M} . By Proposition 1.4 in [BR88] the multiplicity of \mathcal{M} in $L^2(X, \mathcal{B}, \mu)$ is bounded by its dimension, so \mathcal{N} is finite-dimensional. Fix $g \in G$ and put $\mathcal{M}_g = \{S_1^g f : f \in \mathcal{M}\}$. Since S_1 and S_2 commute the representations of G on \mathcal{M} and \mathcal{M}_g determined by S_2 are equivalent. Thus $\mathcal{M}_g \subset \mathcal{N}$. This implies $S_1^g f \in \mathcal{N}$ for all $g \in G$, so f is contained in $\mathcal{E}(S_1)$. \square

Proposition 7.3. *Let S_1 and S_2 be commuting actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . If S_2 is ergodic then $\mathcal{E}(S_2) \subset \mathcal{E}(S_2 S_1)$.*

Proof. Let f be an eigenfunction of S_2 . Form \mathcal{M} and \mathcal{N} as in the proof of Proposition 7.2. Fix $g \in G$. Put $\mathcal{M}_g = S_2^g S_1^g \mathcal{M}$. We have $\mathcal{M}_g = S_1^g \mathcal{M}$, which is equivalent to \mathcal{M} and therefore contained in \mathcal{N} , as desired. \square

We can now give a proof of Bergelson's conjecture when $k = 3$ and the actions T_1, T_2 and $T_2 T_1$ are all ergodic.

Theorem 7.4. *Let T_1, T_2, T_3 be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . Suppose that the actions*

T_1, T_2 and $T_2 T_1$ are ergodic. Then

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} f \cdot T_3^g T_2^g T_1^g f \cdot T_3^g T_2^g f \cdot T_3^g f \, d\mu \, dm(g) > 0 \quad (7.4)$$

for any $f > 0$ in $L^\infty(X, \mathcal{B}, \mu)$.

Proof. Ergodicity of T_1 means $\mathcal{C}_{1,1}$ is trivial, so $\mathcal{C}_{2,1}$ and $\mathcal{C}_{2,2}$ correspond to the functions that are $T_2 T_1$ and T_2 almost-periodic over the trivial factor respectively. Let \mathcal{D} be the sub- σ -algebra of \mathcal{B} corresponding to the functions that are almost-periodic for T_3 over the trivial factor. Combining Corollary 4.8 with Propositions 7.2 and 7.3 gives $\mathcal{C}_{2,2} \subset \mathcal{C}_{2,1} \subset \mathcal{D}$. This implies any $\mathcal{C}_{2,1}$ measurable function f is almost-periodic for both $T_2 T_1$ and T_3 , so $f \in \mathcal{E}(T_3 T_2 T_1)$.

We begin by showing that

$$\liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} T_3^g T_2^g T_1^g f \cdot T_3^g T_2^g f \cdot T_3^g f \cdot f \, dm(g) \, d\mu > 0$$

whenever $f = 1_B$ is $\mathcal{C}_{2,1}$ measurable and $\mu(B) > 0$. Put $\varepsilon = \mu(B)^2/6$. Since $\mathcal{C}_{2,1} \subset \mathcal{D}$ the set Ω_1 of g in G for which $\|T_3^g 1_B - 1_B\| \leq \varepsilon$ and $\|T_3^g T_2^g T_1^g 1_B - 1_B\| \leq \varepsilon$ is a measurable IP^* subset of G . By the argument on page 50 of [Ber96] the set Ω_2 consisting of those g for which $\mu(B \cap (T_3^g T_2^g)^{-1} B) \geq \mu(B)^2 - \varepsilon$ is IP^* . It is also measurable, so the intersection $\Omega = \Omega_1 \cap \Omega_2$ is measurable and IP^* . We have

$$\begin{aligned} & \liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \int_{\Phi_N} T_3^g T_2^g T_1^g 1_B \cdot T_3^g T_2^g 1_B \cdot T_3^g 1_B \cdot 1_B \, d\mu \, dm(g) \\ & \geq \liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} 1_\Omega(g) \left(\int T_3^g T_2^g 1_B \cdot 1_B \, d\mu - 2\varepsilon \right) dm(g) \\ & \geq \liminf_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int_{\Phi_N} 1_\Omega(g) \, dm(g) \cdot (\mu(B)^2 - 3\varepsilon) = \frac{\underline{d}(\Omega) \mu(B)^2}{2} > 0 \end{aligned}$$

because every IP^* set has positive lower density.

The fact that $\mathcal{C}_{2,2} \subset \mathcal{C}_{2,1}$ implies

$$\int f_1 \otimes f_2 \otimes f_3 \, d\nu_2 = \int \mathbb{E}(f_1 | \mathcal{C}_{2,1}) \otimes \mathbb{E}(f_2 | \mathcal{C}_{2,1}) \otimes \mathbb{E}(f_3 | \mathcal{C}_{2,1}) \, d\nu_2$$

for all f_1, f_2, f_3 in $L^\infty(X, \mathcal{B}, \mu)$ where ν_2 is the Furstenberg joining for the actions T_1 and T_2 . Applying Theorem 5.2 with $\mathcal{D}_{2,i} = \mathcal{C}_{2,1}$ gives sub- σ -algebras $\mathcal{E}_{3,i}$ that are characteristic for (7.4). Moreover $\mathcal{E}_{3,i} \rightarrow \mathcal{C}_{2,1}$ is compact for $T_3 \cdots T_i$. Finally, using Theorem 6.1 with $k = 3$, $\mathcal{D} = \mathcal{C}_{2,1}$ and $\mathcal{E}_i = \mathcal{E}_{3,i}$ yields (7.4). \square

Given commuting, measurable actions T_1, \dots, T_k of G on a separated, countably generated probability space (X, \mathcal{B}, μ) define

$$R_k(B) = \{g \in G : \mu(B \cap (T_k^g \cdots T_1^g)^{-1} B \cap \cdots \cap (T_k^g)^{-1} B) > 0\}$$

for any B in \mathcal{B} . We say that a subset R of G is *syndetic* if there is a compact set $F \subset G$ such that $FS = G$. The following result, based on the argument on page 1199 in [BMZ97], shows that $R_k(B)$ is syndetic whenever it has positive lower density with respect to any Følner sequence.

Lemma 7.5. *Let R be a measurable subset of G that has positive lower density with respect to every Følner sequence. Then R is syndetic.*

Proof. Suppose R is not syndetic. Then for every compact subset F of G we have $FR \neq G$. For each $N \in \mathbb{N}$ choose h_N from $G \setminus \Phi_N^{-1}R$. Then $\Phi_N h_N \cap R$ is empty for all N . However, $N \mapsto \Phi_N h_N$ is a left Følner sequence (because the modular function is everywhere positive) so R must have positive lower density with respect to it, giving the desired contradiction. \square

It now follows immediately from Theorems 7.1 and 7.4 that R_2 is always syndetic and that R_3 is syndetic if T_1, T_2 and $T_2 T_1$ are ergodic.

8. FURTHER RESULTS

Recall that a system \mathbf{X} is said to be *weakly mixing* if the only functions almost-periodic over the trivial factor consisting of one point are the constant functions. An immediate consequence of Theorem 5.1 is that the limit as $N \rightarrow \infty$ of (1.3) is constant whenever all of the systems $(X, \mathcal{B}, \mu, T_j \cdots T_i)$ are weakly mixing. This is because all the $\mathcal{C}_{j,i}$ are trivial in that case. In fact, we have a short proof of the following result.

Theorem 8.1 (2.4 in [BR88]). *Let T_1, \dots, T_k be commuting, measurable actions of G on a separated, countably generated probability space (X, \mathcal{B}, μ) . If each of the actions*

$$T_1, T_2 T_1 \times T_2, T_3 T_2 T_1 \times T_3 T_2 \times T_3, \dots, T_k \cdots T_1 \times \cdots \times T_k T_{k-1} \times T_k$$

is ergodic in the corresponding product space $(X^i, \mathcal{B}^i, \mu^i)$ then

$$\lim_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \prod_{i=1}^k T_k^g \cdots T_i^g f_i \, d\mathbf{m}(g) = \prod_{i=1}^k \int f_i \, d\mu$$

for any $f_1, \dots, f_k \in L^\infty(X, \mathcal{B}, \mu)$.

Proof. It suffices to prove that the Furstenberg joining associated to the actions T_1, \dots, T_k is the product measure μ^{k+1} . First note that when $k = 1$ this follows from Proposition 2.2 because \mathcal{C}_1 is trivial by hypothesis.

Let T_1, \dots, T_k be commuting, measurable actions satisfying the above ergodicity assumptions. Assume by induction that ν_{k-1} , the Furstenberg joining for the actions T_1, \dots, T_{k-1} , is the product measure μ^k . We know from Theorem 5.3 that

$$\int f_1 \otimes \cdots \otimes f_{k+1} \, d\nu_k = 0 \tag{8.1}$$

whenever $\mathbb{E}(f_1 \otimes \cdots \otimes f_k | \mathcal{I}_k) = 0$. By hypothesis $T_k \cdots T_1 \times \cdots \times T_k$ is ergodic. Therefore

$$\mathbb{E}(f_1 \otimes \cdots \otimes f_k | \mathcal{I}_k) = \int f_1 \otimes \cdots \otimes f_k \, d\nu_{k-1} = \int f_1 \, d\mu \cdots \int f_k \, d\mu$$

for any f_1, \dots, f_k in $L^\infty(X, \mathcal{B}, \mu)$. Thus (8.1) holds whenever $\int f_i \, d\mu = 0$ for some $1 \leq i \leq k$ as desired. \square

One reason for being interested in Bergelson's conjecture is that it guarantees the existence of certain structures in large subsets of locally-compact, second-countable, amenable groups. To make this precise, one needs to settle on a notion of largeness and then describe a correspondence principle that produces relevant measure-preserving actions from such sets. This has been done in [BF09]. The correspondence principle does not yield *ergodic* measure-preserving actions, so we cannot deduce combinatorial results from Theorem 7.4. However, no ergodicity assumptions were made in the proof of Theorem 7.1, and we now turn to combinatorial consequences of this result, discrete versions of which appear in [BM98] and [BMZ97].

Given an invariant mean M on G , we will say that a subset S of G is *substantial* if one can find a measurable subset W of G with $M(W) > 0$ and a symmetric open neighbourhood U of 1 in G such that $S \supset UW$. Throughout this section we assume G is infinite.

Theorem 8.2. *Given an invariant mean M on G and substantial subsets S_1, \dots, S_k of G one can find $c > 0$, a measurable action T of G on a separated, countably generated probability space (X, \mathcal{B}, μ) , and sets B_1, \dots, B_k in \mathcal{B} with positive measure such that*

$$M(g_1^{-1}S_1 \cap \dots \cap g_k^{-1}S_k) \geq c\mu((T^{g_1})^{-1}B_1 \cap \dots \cap (T^{g_k})^{-1}B_k)$$

for any g_1, \dots, g_k in G .

Proof. The only discrepancies with Theorem 1.1 in [BF09] are that (X, \mathcal{B}, μ) is separated and countably generated, and that the action is measurable. To overcome the first, note that since G is second-countable the space X obtained via the Gelfand representation in the proof of Theorem 1.1 in [BF09] is a compact metric space. Since the action obtained in [BF09] is weakly measurable, using [Ram85] we can assume the action is measurable. \square

Theorem 8.3. *Let M be an invariant mean on $G \times G$ and let S be a substantial subset of $G \times G$. Then*

$$\{g \in G : M(\{(a, b) \in G \times G : (a, b), (a, gb), (ga, gb) \in S\}) > 0\}$$

is syndetic.

Proof. The product $G \times G$ is also a locally-compact, second-countable, amenable group. Let S be a substantial subset of $G \times G$. By the above theorem we can find an action T of $G \times G$ on a separated, countably generated probability space (X, \mathcal{B}, μ) , a set $B \in \mathcal{B}$ having positive measure and some $c > 0$ such that

$$M(S \cap (1, g)^{-1}S \cap (g, g)^{-1}S) \geq c\mu(B \cap (T^{(1, g)})^{-1}B \cap (T^{(g, g)})^{-1}B)$$

for every $g \in G$. Define commuting actions T_1 and T_2 of G on (X, \mathcal{B}, μ) by $T_1^g = T^{(g, 1)}$ and $T_2^g = T^{(1, g)}$. The above becomes

$$M(S \cap (1, g)^{-1}S \cap (g, g)^{-1}S) \geq c\mu(B \cap (T_2^g)^{-1}B \cap (T_1^g)^{-1}B) \quad (8.2)$$

for every $g \in G$. By the discussion at the end of Section 7, the right-hand side of (8.2) is positive for a syndetic set of $g \in G$. \square

For our second result we need some facts about sets of recurrence. A subset S of G is said to be *good for double recurrence* if the intersection of S with

$$\{g \in G : \mu(B \cap (T_2^g T_1^g)^{-1}B \cap (T_2^g)^{-1}B) > 0\} \quad (8.3)$$

contains an element different from the identity for any commuting, measurable actions T_1, T_2 of G on a separated, countably generated probability space (X, \mathcal{B}, μ) and any $B \in \mathcal{B}$ with positive measure.

Lemma 8.4. *If a subset S of G good for double recurrence is finitely partitioned then one of the cells of the partition is good for double recurrence.*

Proof. Fix a partition $S_1 \cup \dots \cup S_r$ of a set S good for double recurrence. Suppose none of the S_i is good for double recurrence. Then for each i one can find a separated, countably generated probability space $(X_i, \mathcal{B}_i, \mu_i)$ equipped with commuting measurable actions $T_{i,1}$ and $T_{i,2}$ such that

$$\mu_i((T_{i,2}^g T_{i,1}^g)^{-1} B_i \cap (T_{i,2}^g)^{-1} B_i \cap B_i) = 0 \quad (8.4)$$

for all $g \neq 1$ in S_i . Let (X, \mathcal{B}, μ) be the product of the above probability spaces and let $B = B_1 \times \dots \times B_r$. Let T_1 and T_2 be the products of the $T_{1,i}$ and the $T_{2,i}$ respectively. Since S is good for double recurrence some S_i contains an element g different from the identity such that

$$0 < \mu((T_2^g T_1^g)^{-1} B \cap (T_2^g)^{-1} B \cap B) = \prod_{i=1}^r \mu_i((T_{i,2}^g T_{i,1}^g)^{-1} B_i \cap (T_{i,2}^g)^{-1} B_i \cap B_i)$$

contradicting (8.4). □

Lemma 8.5. *If S is a measurable subset of G with $\bar{d}(S) = 0$ then $G \setminus S$ is good for double recurrence.*

Proof. First note that $\bar{d}(G) \leq \bar{d}(G \setminus S) + \bar{d}(S)$ so $\bar{d}(G \setminus S) = 1$. Passing to a sub Følner sequence we can assume that $d(G \setminus S) = 1$. Since (8.3) has positive lower density with respect to this Følner sequence it cannot be disjoint from $G \setminus S$. □

Our second result concerns a non-commutative version of Schur's theorem that generalizes the discrete version in [BM98]. Let $Z(g)$ denote the centralizer of g in G . From [BM98] we know that $\{g \in G : [G : Z(g)] < \infty\}$ is a subgroup of G . Moreover, it is measurable because it consists of those points in G that have finite orbit under the action of G on itself by conjugation and hence is a countable union of closed sets. Given a subset C of G denote by \tilde{C} the subset of $G \times G$ consisting of those (a, b) such that ab^{-1} belongs to C .

Lemma 8.6. *For any Følner sequence Ψ in G there is an increasing sequence k_N such that*

$$\frac{m(\Psi_{k_N} \triangle g \Psi_{k_N})}{m(\Psi_{k_N})} \leq \frac{1}{N}$$

for all $g \in \Psi_N$. Moreover, if a measurable subset S of G has density with respect to Ψ_N then \tilde{S} has the same density with respect to $\Psi_N \times \Psi_{k_N}$.

Proof. The first part follows from the fact that $m(\Psi_N \triangle g\Psi_N)/m(\Psi_N)$ converges to 0 uniformly on compact subsets of G . For the second part, if $d(S)$ exists then

$$\begin{aligned} d(S) &= \lim_{N \rightarrow \infty} \frac{m(S \cap \Psi_{k_N})}{m(\Psi_{k_N})} \\ &= \lim_{N \rightarrow \infty} \frac{1}{m(\Psi_N)} \int 1_{\Psi_N}(g) \frac{m(S \cap g\Psi_{k_N})}{m(\Psi_{k_N})} dm(g) \\ &= \lim_{N \rightarrow \infty} \frac{1}{m(\Psi_N)m(\Psi_{k_N})} \iint 1_{\tilde{S}}(g, h) 1_{\Psi_N}(g) 1_{\Psi_{k_N}}(h) dm(h) dm(g) \end{aligned}$$

as desired. \square

Theorem 8.7. *Suppose the subgroup $A = \{g \in G : [G : Z(g)] < \infty\}$ of G does not have finite index. Then for any partition $C_1 \cup \dots \cup C_r$ of G and any open neighborhood U of 1 in G one can find $1 \leq i \leq r$ such that UC_iU contains a subset of the form $\{x, y, xy, yx\}$ with $xy \neq yx$.*

Proof. Fix a partition $C_1 \cup \dots \cup C_r$ of G and an open neighborhood U of 1 in G . Let V be an open neighborhood of 1 such that $VV \subset U$. Permute the indices so that VC_1V, \dots, VC_sV have positive upper density and $VC_{s+1}V, \dots, VC_rV$ have zero upper density with respect to Φ . Since A has infinite index it has zero density. Thus using Lemmas 8.4 and 8.5 we can find $1 \leq i \leq s$ such that $VC_iV \setminus A$ is good for double recurrence. Write $C = VC_iV$. By passing to a sub Følner sequence we can assume that $d(C)$ exists and is positive. By Lemma 8.6 we can find a Følner sequence Ψ in $G \times G$ with respect to which the density of \tilde{C} exists and is positive. If $g \notin A$ then $Z(g)$ has infinite index and thus zero density, so $\tilde{Z}(g)$ will have zero density with respect to Ψ .

Let M be a mean on $G \times G$ that agrees with d_Ψ on the sets having density along Ψ . The set $S = \{(a, b) : ab^{-1} \in UC_iU\}$ contains the substantial subset $(V \times V)\tilde{C}$ of G . Thus by Theorem 8.2 we can find a measurable action T of $G \times G$ on a separated, countably generated probability space (X, \mathcal{B}, μ) , some B in \mathcal{B} with $\mu(B) > 0$ and some $c > 0$ such that

$$M(S \cap (1, g^{-1})S \cap (g^{-1}, g^{-1})S) \geq c \int 1_B \cdot T^{(1, g)} 1_B \cdot T^{(g, g)} 1_B d\mu \quad (8.5)$$

for any $g \in G$. Putting $T_1^g = T^{(g, 1)}$ and $T_2^g = T^{(1, g)}$ we can choose g in $C \setminus A$ such that the right-hand side of (8.5) is positive. Since $M(\tilde{Z}(g)) = 0$ we can find (a, b) in

$$S \cap (1, g^{-1})S \cap (g^{-1}, g^{-1})S \setminus \tilde{Z}(g)$$

giving $\{ab^{-1}, ab^{-1}g^{-1}, gab^{-1}g^{-1}, g\} \subset UC_iU$. Putting $x = ab^{-1}g^{-1}$ and $y = g$ gives the desired result because ab^{-1} does not belong to $Z(g)$. \square

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